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An Alternative Proof of Ptolemy's Theorem and its Variations

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Abstract

This paper introduces a pure geometric proof for Ptolemy's Theorem, without using trigonometry, coordinate geometry, complex numbers, vectors or any other geometric inversion techniques focusing on cyclic quadrilaterals and employing a generalized identity in relation to a cevian of an arbitrary Euclidean plane triangle. Additionally, the paper provides proofs to the converse of Ptolemy's Theorem to which almost no pure geometric complete proof is available, and to the standard Ptolemy's Inequality, to fulfil the research gap in the proofs to some extent. It also includes applications, new corollaries, derived from Ptolemy's Theorem and its converse.

Keywords: Cyclic quadrilaterals, Equilateral triangles, Inequalities, Mathematical logic, Perpendiculars, Similar triangles.

Introduction

The Ptolemy's Theorem of Cyclic Quadrilaterals founded and proved by Claudius Ptolemaeus who was an eminent Greek Mathematician, has been one of the prominent and exciting results in a geometry of a circle, throughout way back centuries ago, even at present not only in Advanced Geometry, but also in the other related sciences. There have been several alternative proofs for the Ptolemy's Theorem of cyclic quadrilaterals in the mathematics literature, using some geometric, trigonometric and non-geometric (Complex number algebra, Vector Algebra) approaches. Amarasinghe (2013) published a concise elementary proof for the Ptolemy's Theorem using purely Euclidean Geometry (without using trigonometry, vector algebra, complex numbers or any other inversion techniques), proving some other useful properties in a cyclic quadrilateral. In this paper, the author adduces an alternative proof for the Ptolemy's Theorem of cyclic quadrilaterals, involving a generalized corollary proved with respect to a cevian of an arbitrary Euclidean triangle covering the cases acute, obtuse, and right triangles, as well as for the converse of the Ptolemy's Theorem involving Mathematical Logic and different Mathematical Proofs.

Main Results

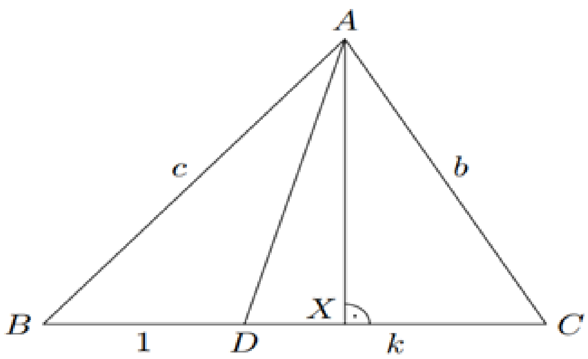
Corollary 1

Let $ABC\Delta$ be an arbitrary plane triangle such that D be an arbitrary point on BC (an internal point), with $BC=a$, $AC=b$ and $AB=c$. If AD is a cevian such that $\frac{BD}{DC}=\frac{1}{k}$ for some $k>0$, then $AD^2=\frac{(1+k)(b^2+kc^2)-a^2k}{(k+1)^2}$

(Amarasinghe, 2011; Amarasinghe, 2012).

Figure 1.

An Euclidean triangle.



Proof of corollary

The proof of the corollary is a conditional proof under proof by cases. For the sake of simplicity (or without loss of generality), assume that $ABC\Delta$ is an acute angle triangle.

Case 1: Assume that AD is not perpendicular to BC.

Proof

Assume that AD is cevian such that $\frac{BD}{DC}=\frac{1}{k}$ Then draw the perpendicular AX to BC. Thus $DX\neq 0$. Using the Pythagoras Theorem respectively for $ABD\Delta$ (Obtuse Triangle), and $ABC\Delta$ (Acute Triangle), it follows that.

$c^2=AD^2-DX^2+(BD+DX)^2=AD^2+BD^2+2BD.DX$, and

$b^2=AD^2-DX^2+(DC-DX)^2=AD^2+DC^2-2DC.DX$

These results lead us to

$\frac{BD}{DC}=\frac{1}{k}=\frac{c^2-AD^2-BD^2}{AD^2+DC^2-b^2}$ since $k>0$ and $DX\neq 0$

Also, it is trivial to see that $BD=\frac{a}{(k+1)}$ and $DC=\frac{ka}{(k+1)}$. Thus, it follows that

$$\frac{1}{k}=\frac{c^2-AD^2-\left(\frac{a}{k+1}\right)^2}{AD^2+\left(\frac{ka}{k+1}\right)^2-b^2}$$

and after some elementary algebraic manipulation, this leads us to the desired

result $AD^2=\frac{(1+k)\left(b^2+kc^2\right)-a^2k}{(k+1)^2}$.

Case 2: Assume that AD is perpendicular to BC. (Now X is coincided with D)

Proof

Then similarly, as before, using the Pythagoras Theorem, it follows $c^2=a^2+b^2-2a.DC$, as well as $b^2=a^2+c^2-2a.BD$.

Thus, it leads to $\frac{BD}{DC}=\frac{1}{k}=\frac{a^2+c^2-b^2}{a^2+b^2-c^2}$

. Thus $k=\frac{a^2+b^2-c^2}{a^2+c^2-b^2}$. Therefore

$k+1=\left(\frac{a^2+b^2-c^2}{a^2+c^2-b^2}\right)+1=\frac{2a^2}{a^2+c^2-b^2}$. Also,

it follows $BD = \dfrac{a^2 + c^2 - b^2}{2a}$.
Then observe that

$$AD^2 = c^2 - BD^2 = c^2 - \left(\frac{a^2 + c^2 - b^2}{2a}\right)^2 = \frac{4a^2c^2 - \left(a^2 + c^2 - b^2\right)^2}{4a^2}$$

Observe that

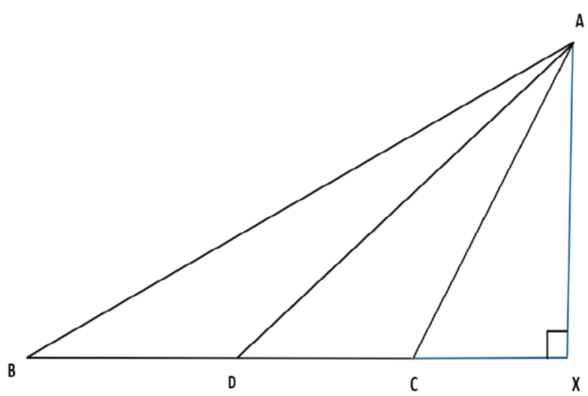
$$\frac{(1+k)(b^2+kc^2)-a^2k}{(k+1)^2} = \frac{\left(\frac{2a^2}{a^2+c^2-b^2}\right)\left(b^2+\left(\frac{a^2+b^2-c^2}{a^2+c^2-b^2}\right)c^2\right)-a^2\left(\frac{a^2+b^2-c^2}{a^2+c^2-b^2}\right)}{\left(\frac{2a^2}{a^2+c^2-b^2}\right)^2}$$

$$= \frac{4a^2c^2-(a^2+c^2-b^2)^2}{4a^2}$$

$= AD^2$.

Hence it follows that in each case,
$AD^2 = \dfrac{(1+k)(b^2+kc^2) - a^2k}{(k+1)^2}$.
Now it is not difficult to prove that, if $ABC\Delta$ is an obtuse triangle or a right-angled triangle, then also $AD^2 = \dfrac{(1+k)(b^2+kc^2) - a^2k}{(k+1)^2}$.
Assume that $ABC\Delta$ is an obtuse triangle. Without loss of generality, assume that the angle $A\hat{C}B$ is an obtuse angle.

Figure 2.
An obtuse Euclidean triangle.



Proof

Assume that AD is cevian such that $\dfrac{BD}{DC} = \dfrac{1}{k}$ for some $k > 0$. Then draw the perpendicular AX to extended BC. Thus $CX \neq 0$. Using the Pythagoras Theorem respectively for $ABD\Delta$ (Obtuse Triangle), and $ADC\Delta$ (Obtuse Triangle), it follows that,

$$c^2 = AD^2 - DX^2 + \left(BD + DX\right)^2 = AD^2 + BD^2 + 2BD.DX = AD^2 + BD^2 + 2BD.(DC + CX)$$

, and hence

$$c^2 = AD^2 + BD^2 + 2BD.(DC + CX) = AD^2 + BD^2 + 2BD.DC + 2BD.CX$$

$$.AD^2 = b^2 - CX^2 + \left(DC + CX\right)^2 = b^2 + DC^2 + 2DC.\,CX$$

These results lead us to

$$\frac{BD}{DC} = \frac{1}{k} = \frac{c^2 - AD^2 - BD^2 - 2BD.DC}{AD^2 - b^2 - DC^2}$$

since $k > 0$ and $CX \neq 0$. Also, it is trivial to see that $BD = \dfrac{a}{k+1}$ and $DC = \dfrac{ka}{k+1}$.

Thus, it follows that

$$\frac{BD}{DC} = \frac{1}{k} = \frac{c^2 - AD^2 - \left(\frac{a}{k+1}\right)^2 - 2\left(\frac{a}{k+1}\right).\left(\frac{ka}{k+1}\right)}{AD^2 - b^2 - \left(\frac{ka}{k+1}\right)^2}$$

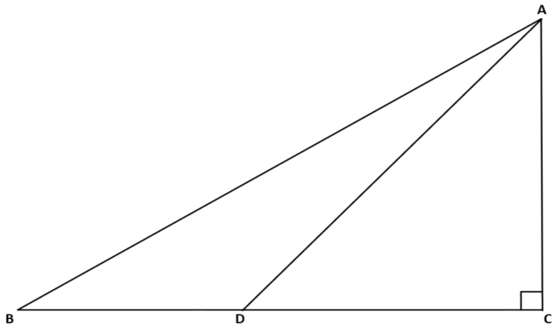
,after some elementary algebraic manipulation, this leads us to the desired result

$$AD^2 = \frac{(1+k)(b^2+kc^2) - a^2k}{(k+1)^2}$$

Assume that $ABC\Delta$ is a right-angled triangle**.** Without loss of generality, assume that the angle $A\hat{C}B$ is right-angle.

Figure 3

A right-angled Euclidean triangle.



Proof

Assume that AD is a cevian such that $\frac{BD}{DC} = \frac{1}{k}$ for some $k > 0$. Then AC is automatically perpendicular to BC . Using the Pythagoras Theorem respectively for $ADC\Delta$ (Right-triangle), it follows that,

$$c^2 = AD^2 = b^2 + DC^2 = b^2 + (BC - BD)^2 = b^2 + \left(a - \frac{a}{k+1}\right)^2$$

since $BD = \frac{a}{k+1}$, and hence this leads us to

$$AD^2 = b^2 + \frac{a^2k^2}{(k+1)^2} = \frac{k^2b^2 + kb^2 + kb^2 + b^2 + a^2k^2}{(k+1)^2} = \frac{k^2(a^2+b^2) + kb^2 + k(c^2 - a^2) + b^2}{(k+1)^2},$$

$$AD^2 = \frac{k^2c^2 + kb^2 + b^2 + kc^2 - a^2k}{(k+1)^2} = \frac{kc^2(k+1) + b^2(k+1) - a^2k}{(k+1)^2}.$$

Thus, this leads us to the required result

$$AD^2 = \frac{(k+1)(b^2 + kc^2) - a^2k}{(k+1)^2}.$$

Observe that now we have proved that for each Euclidean Triangle $ABC\Delta$, the above-mentioned result obtained for the length of the cevian AD, is true.

Theorem 1

Ptolemy's theorem

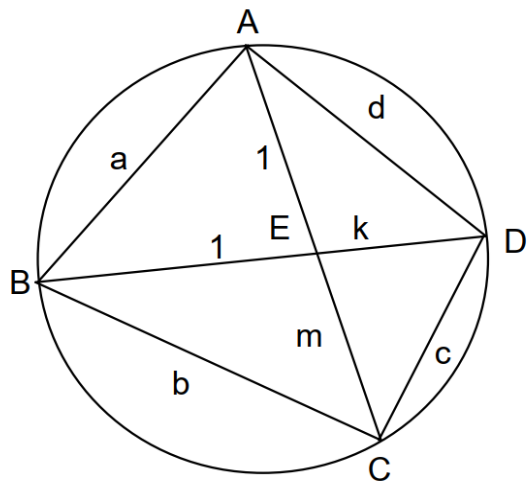
If $ABCD$ is a cyclic quadrilateral such that AC and BD are its diagonals, then $AC.\,BD = AB.\,DC + AD.\,BC$. This is referred to as the Ptolemy's Theorem of Cyclic Quadrilaterals.

Novel proof

Assume that $ABCD$ is a cyclic quadrilateral such that AC and BD are its diagonals. Suppose $AB = a$ $BC = b$, $CD = c$ and $AD = d$. Let E be the point of intersection of the diagonals AC and BD, and let $\frac{BE}{ED} = \frac{1}{k}$ and $\frac{AE}{EC} = \frac{1}{m}$ for some constants $k, m > 0$.

Figure 4.

A cyclic quadrilateral.



Since $B\hat{A}E = E\hat{D}C$, $A\hat{B}E = E\hat{C}D$ and , $ABE\Delta$ and $EDC\Delta$ are similar. Hence $\frac{BE}{EC} = \frac{a}{c}$.

Since $\hat{A}D = E\hat{B}C$, and $A\hat{D}E = E\hat{C}B$, $AED\Delta$ and

$BEC\Delta$ are similar. Hence $\frac{AE}{BE}=\frac{ED}{EC}=\frac{d}{b}$.

Thus $\left(\frac{BE}{EC}\right)\left(\frac{AE}{BE}\right)=\left(\frac{a}{c}\right)\left(\frac{d}{b}\right)$ which leads to $\frac{AE}{EC}=\frac{ad}{bc}=\frac{1}{m}$. Hence $m=\frac{bc}{ad}$.

Also, observe that $\frac{\left(\frac{BE}{EC}\right)}{\left(\frac{ED}{EC}\right)}=\frac{\left(\frac{a}{c}\right)}{\left(\frac{d}{b}\right)}$. Therefore, $\frac{BE}{ED}=\frac{ab}{cd}=\frac{1}{k}$. Hence $k=\frac{cd}{ab}$. Then by using the above corollary on cevians to $ABD\Delta$, we yield $AE^2=\frac{(1+k)\left(d^2+ka^2\right)-BD^2k}{(k+1)^2}$.

Similarly, by using the above corollary to $BCD\Delta$, we yield $EC^2=\frac{(1+k)\left(c^2+kb^2\right)-BD^2k}{(k+1)^2}$.

These two results lead to

$$\frac{AE^2}{EC^2}=\frac{(1+k)\left(d^2+ka^2\right)-BD^2k}{(1+k)\left(c^2+kb^2\right)-BD^2k}=\frac{1}{m^2}$$

By simplifying, this leads to

$$BD^2k\left(m^2-1\right)=(k+1)\left(m^2d^2+m^2a^2k-c^2-kb^2\right)$$

By substituting the above values for k and m, this leads to

$$BD^2\left(\frac{cd}{ab}\right)\left(\left(\frac{bc}{ad}\right)^2-1\right)=\left(\left(\frac{cd}{ab}\right)+1\right)\left(\left(\frac{bc}{ad}\right)^2d^2+\left(\frac{bc}{ad}\right)^2a^2\left(\frac{cd}{ab}\right)-c^2-\left(\frac{cd}{ab}\right)b^2\right)$$

By simplifying we have

$$BD^2(bc-ad)(bc+ad)=(ab+cd)(bd+ac)(bc-ad)$$

Case 1: Assume that $bc\neq ad$. Then it easily follows

$$BD^2=\frac{(ab+cd)(ac+bd)}{(ad+bc)}.$$

It is trivial to see that $AE=\frac{AC}{m+1}$ and $EC=\frac{mAC}{m+1}$. Then observe that

$$AE^2-EC^2=\frac{(1+k)\left(d^2+ka^2\right)-BD^2k}{(k+1)^2}-\left[\frac{(1+k)\left(c^2+kb^2\right)-BD^2k}{(k+1)^2}\right]=\frac{k\left(a^2-b^2\right)+d^2-c^2}{k+1}=$$

$$\left(\frac{AC}{m+1}\right)^2-\left(\frac{mAC}{m+1}\right)^2=\frac{k\left(a^2-b^2\right)+d^2-c^2}{k+1}=AC^2\left(\frac{1-m^2}{(m+1)^2}\right)=AC^2\left(\frac{1-m}{1+m}\right).$$

By substituting the above values for k and m, this leads to

$$AC^2\left(\frac{1-\left(\frac{bc}{ad}\right)}{\left(1+\frac{bc}{ad}\right)}\right)=\frac{\left(\frac{cd}{ab}\right)\left(a^2-b^2\right)+d^2-c^2}{\left(\frac{cd}{ab}\right)+1}$$

. Hence $AC^2\frac{(ad-bc)}{ad+bc}=\frac{(ac+bd)(ad-bc)}{ab+cd}$

Since by our assumption, $bc\neq ad$, it easily follows that $AC^2=\frac{(ad+bc)(ac+bd)}{ab+cd}$.

Case 2: Assume that $bc=ad$.

Then since $m=\frac{bc}{ad}$, it follows $m=1$. That is, then E is the midpoint of AC.

Then by using the Apollonius Theorem for the $ADC\Delta$, it follows that $2AE^2+2ED^2=d^2+c^2$
Observe that by the above-mentioned similar triangles

$$ED=EC\left(\frac{d}{b}\right), \text{ and } EC=\frac{mAC}{m+1},$$

it follows that

$$ED=\left(\frac{mAC}{m+1}\right)\left(\frac{d}{b}\right)=\left(\frac{\left(\frac{bc}{ad}\right)AC}{\left(\frac{bc}{ad}\right)+1}\right)\left(\frac{d}{b}\right)=\frac{AC.\,cd}{bc+ad}$$

Moreover,

$$AE=\frac{AC}{m+1}=\frac{AC}{\left(\frac{bc}{ad}\right)+1}=\frac{ACad}{ad+bc}$$

Thus, by the above Apollonius Theorem, it follows that

$$2\left(\frac{ACad}{ad+bc}\right)^2+2\left(\frac{AC.\,cd}{bc+ad}\right)^2=d^2+c^2$$

By simplifying this further, since $bc=ad$, and rearranging the terms, we yield to the desired result, $AC^2=\frac{(ad+bc)(ac+bd)}{ab+cd}$.

Observe that $EC=BE\left(\frac{c}{a}\right)$. Since $BE=\frac{BD}{k+1}$, it follows $EC=\left(\frac{BD}{k+1}\right)\left(\frac{c}{a}\right)$

Then from the above proved relation, we have

$$EC^2=\frac{(1+k)(c^2+kb^2)-BD^2k}{(k+1)^2}=\left(\frac{BDc}{a(k+1)}\right)^2$$

Substituting for k, we have

$$\frac{BD^2c^2}{a^2\left(\frac{cd}{ab}+1\right)^2}=\frac{\left(1+\frac{cd}{ab}\right)\left(c^2+\left(\frac{cd}{ab}\right)b^2\right)-BD^2\left(\frac{cd}{ab}\right)}{\left(\frac{cd}{ab}+1\right)^2}$$

which leads to $BD^2=\frac{(ab+cd)(ac+bd)}{(ad+bc)}$. That is in each case

$$AC^2=\frac{(ad+bc)(ac+bd)}{ab+cd}$$

and $BD^2=\frac{(ab+cd)(ac+bd)}{(ad+bc)}$.

Hence, we yield

$$AC^2.\,BD^2=\frac{(ad+bc)(ac+bd)}{ab+cd}\times\frac{(ab+cd)(ac+bd)}{(ad+bc)}=(ac+bd)^2$$

Hence, it easily follows $AC.\,BD=AB.\,DC+AD.\,BC$ which is the Ptolemy's Theorem of Cyclic Quadrilaterals. This completes the proposed alternative proof of Ptolemy's Theorem (Alsina & Nelson, 2007; Amarasinghe, 2023).

Remark 1

It also follows that $\frac{AC}{BD}=\frac{ad+bc}{ab+cd}$.

Corollary 2

Assume that $ABCD$ is a cyclic quadrilateral such that AC and BD are its diagonals, and $AB=a$ $BC=b$, $CD=c$ and $AD=d$. Then the intersection point E of the diagonals is the midpoint of AC if and only if $bc=ad$.

Proof of corollary 2

Proof is trivial under the above case 2, if $m=1$.

The converse of the Ptolemy's theorem (converse of Theorem 1)

Let A,B,C and D be four arbitrary points in a plane. If $AC.\,BD = AB.\,DC + AD.\,BC$ such that AC and BD are the diagonals of the quadrilateral $ABCD$, then the points A, B, C and D are on a circle.

Novel proof

Proof is a proof by contraposition and proof by cases. Assume that at least one point of A, B, C and D is not on a circle. Without loss of generality, assume that D is not on the circle.

Figure 5.

D is outside the circle.

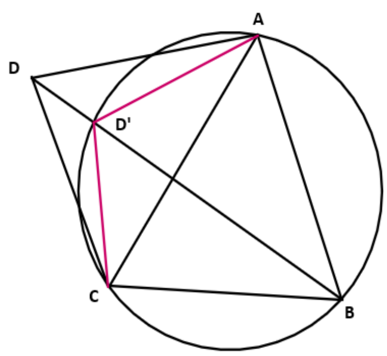


Figure 6.

D is inside the circle.

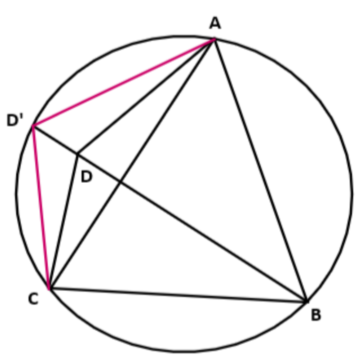
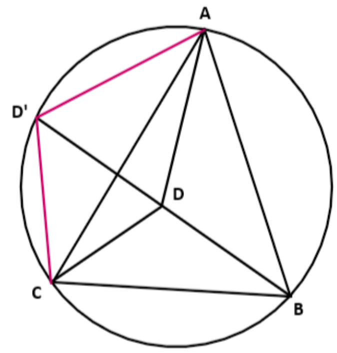


Figure 7.

D is inside $ABC\Delta$.



Case 1: Assume that D is outside the $ABC\Delta$ and the circumcircle of $ABC\Delta$ (Fig. 5).

Proof

Using the Ptolemy's Theorem, it follows $AC.\,BD' = AB.\,D'C + AD'.\,BC$ Since $A\hat{D}'D > A\hat{D}D'$ is an obtuse angle, by the very elementary geometry, it is trivial to see that . Thus $AD > AD'$. Similarly, it follows $CD > CD'$. Also, $BD > BD'$. Hence $AB.CD + BC.AD > AB.\,D'C + AD'.\,BC = AC.BD'$. That is, $\frac{AB.CD + BC.AD}{AC} > BD'$.

Therefore, by writing $BD' = BD - DD'$, it follows that

$$DD' > BD - \left[\frac{AB.\,CD + BC.AD}{AC}\right]$$

which easily leads to

$$DD' > \frac{AC.BD - AB.CD - BC.AD}{AC}.$$

Now carefully observe that since the point D is an arbitrary external point outside the circle, the length $DD' > 0$ is arbitrary.

Due to the arbitrariness of $DD' > 0$, by using a well-established corollary in measure theory (or in real analysis), it follows that $AC.\,BD \leq AB.\,DC + AD.\,BC$ That is, the compound statement "or $AC.\,BD = AB.\,DC + AD.\,BC$" is true including two cases.

Now we claim that the case $AC.\,BD = AB.\,DC + AD.\,BC$ is false. On the contrary assume that $AC.\,BD = AB.\,DC + AD.\,BC$ is true. Then Since by Ptolemy's theorem $AC.\,BD' = AB.\,D'C + AD'.\,BC$, it follows that,

$$AB.\,D'C + AD'.\,BC + AC.DD' = AD.BC + AB.CD$$

which leads us to,

$$AB.\left(CD' - CD\right) + BC\left(AD' - AD\right) + AC.DD' = 0$$

Observe that according to our choice, AB, BC and AC are fixed non-zero sides, and thereby it follows that $CD' - CD = 0$, $AD' - AD = 0$ and *in particular* $DD' = 0$. This is a contradiction since D is an outside point of the circle with $DD' > 0$ under our assumption in this case. This proves that it is not the case that $AC.\,BD = AB.\,DC + AD.\,BC$.

Then it follows that only the case $AC.\,BD < AB.\,DC + AD.\,BC$ is true. Thus $AC.\,BD \neq AB.\,DC + AD.\,BC$. Thus, by contraposition, the converse of the Ptolemy's Theorem is proved.

Case 2: Assume that D is outside the $ABC\Delta$ but is inside the circumcircle of $ABC\Delta$ (Fig.6)

Proof

Using the Ptolemy's Theorem, it follows $AC.\,BD' = AB.\,D'C + AD'.\,BC$. Similarly, as in case 1, by using the very elementary geometry, it follows that $AD < AD'$, $CD < CD'$ and $BD < BD'$. In addition, $BD' = BD + DD'$. Due to the arbitrariness of $DD' > 0$, as proved in the case 1, by using the related corollary in measure theory and proving $AC.\,BD \neq AB.\,DC + AD.\,BC$ as before, this leads us to $AC.\,BD < AB.\,DC + AD.\,BC$, that is, $AC.\,BD \neq AB.\,DC + AD.\,BC$.

Thus, by contraposition, the converse of the Ptolemy's Theorem is proved.

Case 3: Assume that D is inside the $ABC\Delta$, and inside the circumcircle of $ABC\Delta$ (Fig. 7)

Proof

Using the Ptolemy's Theorem, it follows $AC.\,BD' = AB.\,D'C + AD'.\,BC$. In this case it is possible that $AD = AD'$ and $CD = CD'$, or $AD < AD'$ and $CD < CD'$, or $AD > AD'$ and $CD > CD'$. But since $BD < BD'$, even if $AD = AD'$ and $CD = CD'$, it follows that $AC.\,BD < AB.\,DC + AD.\,BC$ similarly, as we proved in above cases 1 and 2. In the rest of the cases of $AD \neq AD'$ and $CD \neq CD'$, similarly, it follows that $AC.\,BD < AB.\,DC + AD.\,BC$ as in the above case 2 and 1. Thus, in all possible cases, it follows that $AC.\,BD \neq AB.\,DC + AD.\,BC$. Thus, by contraposition, the converse of the Ptolemy's Theorem is proved.

Remark 2

Observe the fact that the author could have given the proof for the converse of the Ptolemy's Theorem only by proving it is not the case that $AC.\,BD = AB.\,DC + AD.\,BC$ in each of the above mentioned 3 cases as proved in the proof of the latter part of case 1. But if so then it could not have been led to the proof of the following well-established inequality.

Corollary 3

Ptolemy's Inequality

Let A, B, C and D be four arbitrary points in a plane with the same orientation. Then $AC.\,BD \le AB.\,DC + AD.\,BC$.

Novel proof of corollary 3

Initially assume that $ABCD$ is a convex (Fig. 5 and Fig. 6) or concave (Fig. 7) quadrilateral. Then due to the orientation, AC and BD are its diagonals. Observe that in all the above cases 1, 2 and 3 under the proof of the converse of the Ptolemy's Theorem, we have proved $AC.\,BD < AB.\,DC + AD.\,BC$ by proving $AC.\,BD \le AB.\,DC + AD.\,BC$ initially and disproving the equality (since $DD' \neq 0$) with the assumption that D is an outside or inside point of the circle. Now assume that D is an arbitrary point that is on or outside or inside the circle. Observe that in the case of a concave quadrilateral, it falls under case 3 (Fig. 7), wherein point D is inside the $ABC\Delta$.

Hence from the proofs of cases 1, 2 and 3 of the converses of the Ptolemy's Theorem, it trivially follows that $AC.\,BD \le AB.\,DC + AD.\,BC$ since we assume that D can also be on the circle. This inequality is called the "**Ptolemy's Inequality**", and the equality follows from Ptolemy's Theorem, in the case of the points A, B, C and D are on the circle.

Nevertheless, note that we have not yet covered the cases, "three points are on a straight line, and the remaining point is outside the straight line", and "all the four points A, B, C and D are on a straight line".

Now assume that three points are on a straight line, and the remaining point is outside the straight line. Without loss of generality, assume that A is outside the straight line BCD.

Figure 8.

Four points form a triangle.

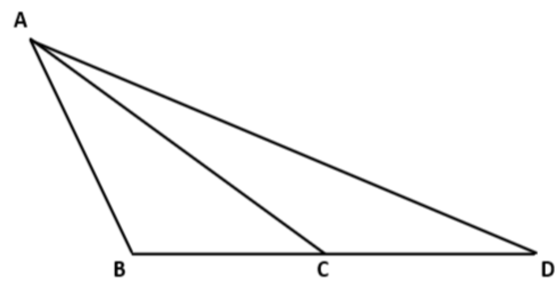
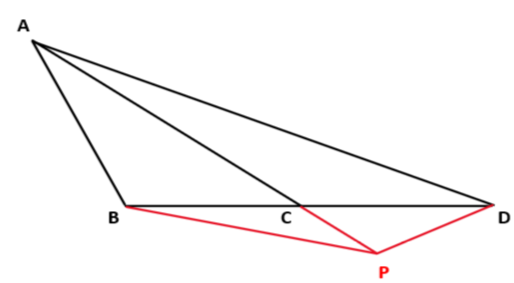


Figure 9.

P is at near vicinity of C.



Let P be an arbitrary point in the near vicinity of the point C such that ACP is a straight line (**Fig. 9**). Now since $ABPD$ is a quadrilateral, it follows from the above result that $AP.\,BD \le AB.\,DP + AD.\,BP$. Observe that $\lim_{P\to C} AP$, $\lim_{P\to C} BP$ and $\lim_{P\to C} DP$ do exist as real numbers. Hence it is not difficult to prove by using real analysis that,

$$\lim_{P\to C} AP.\,BD \le \lim_{P\to C}\left(AB.\,DP + AD.\,BP\right)$$

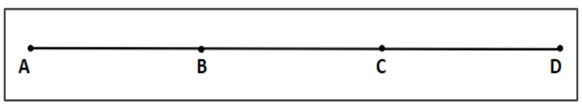
since BD, AB and AD are fixed relative to the point P. That is $BD.\lim_{P\to C} AP \le AB.\lim_{P\to C} DP + AD.\lim_{P\to C} BP$ which leads us to,

$$AC.\,BD \le AB.\,DC + AD.\,BC.$$

Now assume that the points A, B, C and D are on a straight line.

Figure 10.

Straight line ABCD.



Then observe that

$$AC.\,BD = (AB + BC)(BC + CD) = AB.\,BC + AB.CD + BC^2 + BC.CD$$

Simplifying this leads to

$$AC.\,BD = BC.(AB + BC + CD) + AB.CD = AD.BC + AB.CD$$

which is the equality in Ptolemy's Inequality. Thus in all the possible arrangements of arbitrary points A, B, C and D with the same orientation, it follows that $AC.\,BD \le AB.\,DC + AD.\,BC$. This completes the proof of Ptolemy's Inequality (Apostol, 1967).

Remark 3

Observe that the **equality** in Ptolemy's Inequality appears, if $ABCD$ is a Cyclic Quadrilateral **or** $ABCD$ is a straight line.

Corollary 4

Under the converse of Ptolemy's Theorem

Let $ABC\Delta$ is an equilateral triangle in a plane. Assume that the point D is outside the $ABC\Delta$ being on the same plane such that AC and BD are the diagonals of the quadrilateral $ABCD$ with $BD = AD + DC$. Then the points A, B, C and D are on a circle.

Proof

Since $ABC\Delta$ is an equilateral triangle, it follows that $AB = BC = AC$. Also, since it is given that $BD = AD + DC$, it follows $AC.\,BD = BC.\,AD + AB.\,DC$. Hence, by the converse of the Ptolemy's Theorem, it follows that the points A, B, C and D are on a circle.

Remark 4

Observe that the converse of the above Corollary 4 is a very well-established result in circle geometry (Aliyev et al., 2020).

Conclusions

In this paper, the Ptolemy's Theorem of Cyclic Quadrilaterals is proved by a different approach using a derived identity around a cevian of a triangle though the author himself has already given a shorter proof of the same theorem (Amarasinghe, 2013). The new approach has many advantages. Thus, the readers are encouraged to analyse the author's novel approach of the proof of the

Ptolemy's Theorem presented here, as it leads to many other significant and important new corollaries in being attempted to prove the Ptolemy's Theorem in this way. Moreover, the converse of the Ptolemy's Theorem along with the standard Ptolemy's Inequality proved by using the contraposition and proof by cases, is also important since it is hard to find complete proofs for the converse of the Ptolemy's Theorem in a Euclidean Geometric way.

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